Characterization of Variation Averse Preferences by Present Value

Umut Keskin [1]

Abstract

As a model for intertemporal decision making, Gilboa (1989) introduced variation aversion and provided its characterization through preference conditions. In this paper, it is shown that the same model can easily be characterized by conditions based on present value. Being a familiar and intuitive concept, present value framework simpli es the existing characterization. Our result also makes the model testable by econometric methods.

Keywords: Present Value, Intertemporal Choice, Discounted Utility, Variation Aversion.

JEL Codes : C02, D01, D03.

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Characterization of Variation Averse Preferences by Present Value

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Abstract

As a model for intertemporal decision making, Gilboa (1989) introduced variation aversion and provided its characterization through preference conditions. In this paper, it is shown that the same model can easily be characterized by conditions based on present value. Being a familiar and intuitive concept, present value framework simplifies the existing characterization. Our result also makes the model testable by econometric methods.

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1 Introduction

Since its introduction by Samuelson (1937), the constant discounted utility (CDU) function has been the most prominent tool for analyzing intertemporal choice in economics. CDU model is additively separable which is analytically a powerful property but sometimes violated as in the variation averse preferences in Gilboa (1989). In that model, individuals have distaste for fluctuations in future income streams, therefore preferences are not separable. We will use the present value (PV) framework of Bleichrodt et al. (2015) to give a characterization of Gilboa's alternative model to CDU. In his original work, Gilboa (1989) already gave the characterization of variation aversion model which was based on preference conditions. However, our

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new characterization will not be yet another characterization of an existing model. First, our PV characterization will make the model more easily testable since experimental studies in intertemporal choice are inevitably carried out from today's perspective and PV best fits this perspective. Moreover, our result has an advantage that the original characterization did not posses: the model can now be tested by using econometric methods, regressing the PV on relevant variables described below. And finally, the PV conditions in our result better reflect the essence of the model than the earlier characterization. We will show that variation averse agents are exactly those who take into consideration the fluctuations of income between periods when assessing changes in their future incomes. We find such consideration quite reasonable and therefore, our main result suggests that additive separability is hard to defend in the intertemporal setting, making the most widely used model in economics in this setting questionable from a normative perspective. Hence we intend to shed some light on the normative discussions on the CDU model.

The next section introduces the PV framework. Section 3 describes Gilboa's (1989) model and result. In section 4, we present and discuss our main result. Section 5 concludes. Proof of our main result is in the Appendix.

2 Present Value

Let $S = \{0, \ldots, n\}$ be the set of time periods with $n \geq 2$, where 0 is the current time. At each period i, an individual receives a monetary outcome $x_i \in \mathbb{R}$. Agents are assumed to have a continuous, complete, strictly monotone and transitive preference relation $\succsim$ on the set of all income streams, $\mathbb{R}^{n+1}$. $\succ$ and $\sim$ are defined as usual. PV of an individual is formally defined as follows, as first introduced by Bleichrodt et al. (2015):

Definition 2.1. *Suppose that an individual with preference relation $\succsim$ is endowed with $x \in \mathbb{R}^{n+1}$. Let x change by ϕ units in period i. Then π is called the* ***present value*** *of ϕ if:*

$$(x_0 + \pi, x_1, \ldots x_i, \ldots, x_n) \sim (x_0, \ldots, x_i + \phi, \ldots, x_n).$$

For simplicity, we will assume that for all $x, y \in \mathbb{R}^{n+1}$ with $x \nsim y$ and all $i \in \{0, \ldots, n\}$, there exists $\varepsilon_i \in \mathbb{R}^{n+1}$ such that $(x_0, \ldots, x_i + \varepsilon_i, \ldots, x_n) \sim y$ which guarantees the existence of π[1]. Strict monotonicity implies the uniqueness of π. PV may depend on i, ϕ, x or other forms depending on the decision model used. Such dependence is denoted by $\pi_i(\phi, x, \cdot)$. We abide

[1]See Bleichrodt et al. (2015) for a generalization

by the clarity of mathematical language and omit any irrelevant variable in the notation for π. Clearly, $\pi_0(\phi, x, \cdot) = \phi$.

In the commonly used CDU model, preferences are represented by $\sum_{i=0}^{n} \beta^i u(x_i)$ where, $0 < \beta < 1$ discounts future outcomes and u is a real valued function on monetary payoffs. The model below by Gilboa (1989) constitutes a case where where this functional form is violated.

3 Variation Aversion

Consider the following example (Gilboa (1989)): In a four period setting, let H denote high payment and L denote low payment. Assume that an agent dislikes variation in her payments and we observe the preferences below:

$$(H, H, L, L) \sim (L, L, H, H) \succ (H, L, H, L) \sim (L, H, L, H) \tag{1}$$

There is less variation in the first two streams than in the latter two, hence the preferences above (see Gilboa (1989) for a more detailed justification of such preferences). It can be seen from (1) that the CDU model, or in fact any additively separable form cannot accommodate these preferences. We will analyze the variation aversion model (Gilboa (1989)) that can explain (1).

As the underlying reasoning suggests, any attempt to model preferences in (1) should take into account the variation in utility terms between periods, $|u(x_i) - u(x_{i-1})|$. Gilboa (1989) introduced a preference condition, the *variation preserving sure thing principle* (VPSTP), which leads to the following representation:

$$\sum_{i=0}^{n} (\lambda_i u(x_i) + \tau_i |u(x_i) - u(x_{i-1})|). \tag{2}$$

In (2), discounted utility is adjusted by a weighted sum of utility variations in each period. Once these variations are incorporated, (2) can explain the distaste for fluctuations in income.

Let $x_{-1} = x_{n+1} = 0$ and call subsets of S of the form $\{i, \ldots, j\}$ for $i \leq j$ *intervals*, which are denoted by $[i, j]$.

Definition 3.1. *Let $A = [i, j] \subset S$ be an interval and let $x, x', y, y' \in \mathbb{R}^{n+1}$ be such that*

$$\begin{aligned} x_k = y_k, \qquad & x'_k = y'_k \qquad \forall k \in A \\ x_k = x'_k, \qquad & y_k = y'_k \qquad \forall k \in A^{\mathsf{c}} \\ x_k = x'_k = y_k = y'_k \quad & \text{for} \quad k = i-1, \ j+1. \end{aligned}$$

Then $\succsim$ is said to satisfy variation preserving sure thing principle if $x \succsim y$ if and only if $x' \succsim y'$.

Gilboa (1989) gave the following characterization of (2).

Theorem 3.1. *(Gilboa 1989) The following statements are equivalent:*

(i) $\succsim$ *is represented by*

$$\sum_{i=0}^{n}(\lambda_i u(x_i) + \tau_i |u(x_i) - u(x_{i-1})|)$$

for all $x \in \mathbb{R}$*, where* $u : \mathbb{R} \to \mathbb{R}$ *is unique up to an increasing affine transformation and*

$$\begin{aligned} |\lambda_i| &\geq |\tau_i| + |\tau_{i+1}| \quad \forall i < n \\ |\lambda_n| &\geq |\tau_n| \text{ and } \tau_0 = 0. \end{aligned}$$

(ii) $\succsim$ *satisfies VPSTP.*

In the above theorem, u is the utility function for monetary outcomes. Since it is unique up to an increasing affine transformation, we can rescale it and get $u(0) = 0$.

4 Main Result

Instead of characterizing (2) by VPSTP, we now present a simpler and empirically more operational axiomatization in terms of present value behavior. For the result below, we will further assume[2] that $u \in \mathcal{C}^\infty$. Let $\Delta(\alpha, \beta)$ denote the difference $\alpha - \beta$.

Proposition 4.1. *The following statements are equivalent:*

(i) $\succsim$ *is represented by*

$$\sum_{i=0}^{n}(\lambda_i u(x_i) + \tau_i |u(x_i) - u(x_{i-1})|)$$

for all $x \in \mathbb{R}$*, where* $u : \mathbb{R} \to \mathbb{R}$ *is unique up to an increasing affine transformation and*

$$\begin{aligned} |\lambda_i| &\geq |\tau_i| + |\tau_{i+1}| \quad \forall i < n \\ |\lambda_n| &\geq |\tau_n| \text{ and } \tau_0 = 0. \end{aligned}$$

(ii) The PV depends on $\phi, i, x_0, x_i, \Delta(x_1, x_0), \Delta(x_i, x_{i-1})$ *and* $\Delta(x_{i+1}, x_i)$*:*

$$\pi = \pi_i(\phi, x_0, x_i, \Delta(x_1, x_0), \Delta(x_i, x_{i-1}), \Delta(x_{i+1}, x_i)).$$

Proof. See Appendix. □

[2] Debreu (1972) gives behavioral conditions on differentiability.

For the agent described above, variation between period-0 and 1 incomes, and the variations between period-i income and the incomes right before and right it affect the perception of PV and this is all that is needed to have (2). Since this variation idea is the very essence of the model we analyze; compared to Definition 3.1, the PV characterization is more intuitive and clarifies the rationale behind the model. As discussed in Section 2, the PV notation we use associates π only with the relevant variables for it. Aside from the mathematical aesthetic, Proposition 4.1 shows that such notation proves to represent the complete nature of the model in hand. This compactification of the variation aversion model to a single notation is the very novelty of our paper. This also makes the model statistically testable in empirical studies if we run a regression of π over its explanatory variables. Also, since evaluation of future income changes in consideration with neighboring income levels is quite a reasonable attitude, we find additive separability hard to defend normatively. This makes the CDU model questionable, so our result we anticipate to motivate more research on this issue.

5 Conclusion

We presented a present value foundation for a model of intertemporal choice, variation aversion. Although our result is mainly analytical, our goal is to provide a natural road map for experimental researchers who wish to test the validity of intertemporal decision models. In this respect, we presented an easily testable condition in terms of subjective PV which is based on the main idea of the model.

Appendix: Proof of Proposition 4.1

Suppose that (i) holds. The case $i = 0$ is trivial. Then for $1 \leq i < n$, Definition 2.1 implies

$$\lambda_0 u(x_0 + \pi) + \tau_1 |u(x_1) - u(x_0 + \pi)| + \lambda_i u(x_i) + \tau_i |u(x_i) - u(x_{i-1})| + \tau_{i+1} |u(x_{i+1}) - u(x_i)| =$$
$$\lambda_0 u(x_0) + \tau_1 |u(x_1) - u(x_0)| + \lambda_i u(x_i + \phi) + \tau_i |u(x_i + \phi) - u(x_{i-1})| + \tau_{i+1} |u(x_{i+1}) - u(x_i + \phi)| \tag{A.1}$$

Showing that (ii) holds is a matter of solving A.1 for π for different cases. This arithmetic is not crucial for our purposes, so we will present the result for only one illustrative case:

$$x_0 \geq 0, \quad \phi > 0, \quad \Delta(x_i, x_{i-1}) \geq 0, \quad \Delta(x_{i+1}, x_i) \geq \phi, \text{and} \tag{A.2}$$
$$\frac{D_i}{D_0}(u(x_i + \phi) - u(x_i)) \leq u(x_1) - u(x_0) \tag{A.3}$$

where $D_i = \lambda_i + \tau_i - \tau_{i+1}$ and $D_0 = \lambda_0 - \tau_1$. (1) implies $D_0 u(x_0 + \pi) + D_i u(x_i) = D_0 u(x_0) + D_i u(x_i + \phi)$. Note that, $D_0 \neq 0$, for otherwise one would have $D_i = 0$ and this would give infinitely many values of π; which violates monotonicity. Hence, D_i/D_0 is well defined. Then with the assumptions made in A.2 and A.3, we have:

$$\pi = u^{-1}(u(x_0) + \frac{D_i}{D_0}[u(x_i + \phi) - u(x_i)]) - x_0. \tag{A.4}$$

In A.4, π depends on $x_0,\ x_i$ and ϕ explicitly, and on i through D_i in the formula. Dependence on $\Delta(x_{i+1}, x_i)$ and $\Delta(x_i, x_{i-1})$ is due to the conditions assumed in A.2. In A.3, we see dependence of π on $u(x_1) - u(x_0)$. Using Taylor expansion, we modify this as

$$u(x_1) - u(x_0) = \sum_{j=1}^{\infty} \frac{u^{(j)}(x_0)}{j!}[\Delta(x_1, x_0)]^j,$$

where, $u^{(j)}(.)$ is the j^{th} derivative of u. Hence π depends on $\Delta(x_1, x_0)$. So the explanatory variables for π are $\phi,\ x_0,\ x_i,\ \Delta(x_1, x_0),\ \Delta(x_i, x_{i-1})$ and $\Delta(x_{i+1}, x_i)$, and nothing else. The case for $i = n$ is investigated in the same manner, only noting that $x_{n+1} = 0$. Other possible variations of A.2 and A.3 are proved similarly. Therefore,

$$\pi = \pi_i(\phi, x_0, x_i, \Delta(x_1, x_0), \Delta(x_i, x_{i-1}), \Delta(x_{i+1}, x_i)).$$

Conversely, assume that (ii) holds. We will show that $\succsim$ satisfies VPSTP which will imply that it can be represented by (2). Let $A = [i, j]$ be an interval and $x, x', y, y' \in \mathbb{R}^{n+1}$ be as in Definition 3.1. Assume that $j + 1 < n$ and suppose $x \succsim y$. First we will consider indifference, $x \sim y$. We will get period-0 equivalents of future payments one by one. For this purpose, let us define the recursive sequence p_k as follows:

$$\begin{aligned}
p_0 &= x_0 \\
p_1 &= p_0 + \pi_n(x_n, p_0, 0, \Delta(x_1, p_0), \Delta(0, x_{n-1}), \Delta(x_{n+1}, 0)) \\
&\vdots \\
p_k &= p_{k-1} + \pi_{n+1-k}(x_{n+1-k}, p_{k-1}, 0, \Delta(x_1, p_{k-1}), \Delta(0, x_{n-k}), \Delta(0, 0)). \\
&\vdots
\end{aligned}$$

for $0 \leq k \leq n - (j + 2)$. Also, let

$$\begin{aligned}
\tilde{p}_1 &= p_{n-(j+2)} + \pi_{i-2}(x_{i-2}, p_{n-(j+2)}, 0, \Delta(x_1, p_{n-(j+2)}), \Delta(0, x_{i-3}), \Delta(x_{i-1}, 0)) \\
\tilde{p}_2 &= \tilde{p}_1 + \pi_{i-3}(x_{i-3}, \tilde{p}_1, 0, \Delta(x_1, \tilde{p}_1), \Delta(0, x_{i-4}), \Delta(0, 0)) \\
&\vdots \\
\tilde{p}_{i-2} &= \tilde{p}_{i-3} + \pi_1(x_1, \tilde{p}_{i-3}, 0, \Delta(0, \tilde{p}_{i-3})\Delta(\tilde{p}_{i-3}, 0)\Delta(0, 0)).
\end{aligned}$$

We apply the same procedure to y and will call the resulting values from this, p' and $\tilde{p}'$. Let $\tilde{p}_{i-2} = \Pi$ and $\tilde{p}'_{i-2} = \Pi'$. Using the definition of PV, we get

$$x \sim (p_1, x_1, \ldots, x_{n-1}, 0) \sim (p_2, x_1, \ldots, x_{n-2}, 0, 0).$$

Repeating the above process gives, for $k \in A^{\mathsf{c}} \setminus \{0, i-1, j+1, j+2\}$,

$$x \sim (\Pi, 0, \ldots, 0, x_{i-1}, \ldots, x_{j+2}, 0 \ldots, 0) \tag{A.5}$$

$$y \sim (\Pi', 0, \ldots, 0, y_{i-1}, \ldots, y_{j+2}, 0 \ldots, 0). \tag{A.6}$$

Writing $\Pi = \Pi' + (\Pi - \Pi')$ and $y_{j+2} = x_{j+2} + (y_{j+2} - x_{j+2})$, we have:

$$(\Pi' + (\Pi - \Pi'), 0, \ldots, 0, x_{i-1}, \ldots, x_{j+2}, 0 \ldots, 0) \sim$$
$$(\Pi', 0, \ldots, 0, y_{i-1}, \ldots, y_{j+1}, x_{j+2} + (y_{j+2} - x_{j+2}), 0 \ldots, 0).$$

Since $y_k = x_k$ for $k = i-1, \ldots, j+1$, we have $(\Pi' + (\Pi - \Pi'), 0, \ldots, 0, x_{i-1}, \ldots, x_{j+2}, 0 \ldots, 0)$ $\sim (\Pi', 0, \ldots, 0, x_{i-1}, \ldots, x_{j+1}, x_{j+2} + (y_{j+2} - x_{j+2}), 0 \ldots, 0)$. By the definition of PV, this means

$$\pi_{j+2}(y_{j+2} - x_{j+2}, \Pi', 0, x_{j+1}, x_{j+2}, 0) = \Pi - \Pi'.$$

Since $\pi_{j+2}(.)$ is independent of values between i and j, we have:

$$(\Pi' + (\Pi - \Pi'), 0, \ldots, 0, x_{i-1}, x'_i, \ldots, x'_j, x_{j+1}, x_{j+2}, 0 \ldots, 0) \sim$$
$$(\Pi', 0, \ldots, 0, x_{i-1}, x'_i, \ldots, x'_j, x_{j+1}, x_{j+2} + (y_{j+2} - x_{j+2}), 0 \ldots, 0). \text{ and}$$
$$(\Pi' + (\Pi - \Pi'), 0, \ldots, 0, x_{i-1}, x'_i, \ldots, x'_j, x_{j+1}, x_{j+2}, 0 \ldots, 0) \sim$$
$$(\Pi', 0, \ldots, 0, y_{i-1}, y'_i, \ldots, y'_j, y_{j+1}, x_{j+2} + (y_{j+2} - x_{j+2}), 0 \ldots, 0).$$

Now reverse these operations and forward values back to their initial positions, which gives

$$(\Pi' + (\Pi - \Pi'), 0, \ldots, 0, x_{i-1}, x'_i, \ldots, x'_j, x_{j+1}, x_{j+2}, 0 \ldots, 0) \sim x' \text{ and}$$
$$(\Pi', 0, \ldots, 0, y_{i-1}, y'_i, \ldots, y'_j, y_{j+1}, x_{j+2} + (y_{j+2} - x_{j+2}), 0 \ldots, 0) \sim y'.$$

Hence, $x' \sim y'$ as we wanted to show. If $x \succ y$, then by continuity we can find $\varepsilon > 0$, such that $x \sim (y_0 + \varepsilon, y_1, \ldots, y_n)$. Afterwards, the same argument above is repeated to show that $x' \succ y'$. If $j + 1 = n$, then we start rolling x_k and y_k values back to present time from period $i - 2$ to obtain Π and Π'. Once Π and Π' are defined as such, the rest of the proof follows the same steps as above, and once again we obtain $x' \succsim y'$. This completes the proof.

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